THE FINITE ELEMENT METHOD

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1. INTRODUCTION

Starting 1996, the Laboratory of Structural Mechanics works on the development of a Finite Element Method Software. This software is called “X-Fem”. X-Fem is a software capable on solving linear and nonlinear problems of Structural Mechanics.

Non linearity on problems of structural mechanics consists of material non-linearity and/or geometry nonlinearity. Often nonlinear behavior of a structural system leads to *Instability*. Instability is used as a mathematical term as well as a term of mechanics. As a mathematical term instability leads to singularity or failure to formulate a closed solution. As a term of mechanics instability is often used to describe a mode of *failure*.

Finite Element Method (FEM) is a numerical method for solving partial differential equations. The concept of FEM is to discretize the field to k discrete elements with simple and normal shapes. The field consists of n degrees of freedom (DOF). In Structural Mechanics DOF represent the virtual displacements on points (nodes) where the (finite) elements are constructed. The solution in FEM is succeeded by the solution of a linear system of n equation. On linear problems the solution is:

$[\boldsymbol{K}].\boldsymbol{u}=\boldsymbol{F}$ (1)

$\boldsymbol{u}$ is the vector of displacements, $\boldsymbol{u}=[u_1,u_2,...,u_n]$, $\boldsymbol{F}$ is the vector of loadings $\boldsymbol{F}=[F_1,F_2,...,F_n]$ and $[\boldsymbol{K}]$ is a nxn matrix, called s*tiffness matrix.* Stiffness matrix is representing the response of the field for a given vector of loadings $\boldsymbol{F}$.

In the case of linear behavior, $[\boldsymbol{K}]$ is a linear factor of $\boldsymbol{u}$, independent of the stress status and constant. The stress status is described by the stress tensor $\underline{\boldsymbol{\sigma}}$, $\underline{\boldsymbol{\sigma}}=\boldsymbol{\sigma}(\boldsymbol{u})$. For the case of nonlinear behavior either is derived by material nonlinearity or either is derived by geometry nonlinearity (large deformations) the $[\boldsymbol{K}]$ is expressed as a function of the stress tensor: $[\boldsymbol{K}]=[\boldsymbol{K}(\underline{\boldsymbol{\sigma}})]$, so the eq. (1) is not valid. To succeed a solution for nonlinear problems, vector $\boldsymbol{F}$ is divided on i stages $\delta\boldsymbol{F}$, called *increments*. The eq. (1) becomes then:

$[\boldsymbol{K}ep].\delta\boldsymbol{u}=\delta\boldsymbol{F}+\boldsymbol{R}_{i-0}$ (2) and
$\underline{\boldsymbol{\sigma}}_i=\underline{\boldsymbol{\sigma}}_{i-1}+\boldsymbol{\sigma}(\delta\boldsymbol{u})$ (2a)

$[\boldsymbol{K}ep]$ is a linear factor and it is equal with $[\boldsymbol{K}(\underline{\boldsymbol{\sigma}})]$ depending on the $\underline{\boldsymbol{\sigma}}$ (stress tensor) of the current increment, $\delta\boldsymbol{u}$ are the displacements induced by the increment $\delta\boldsymbol{F}$. $\boldsymbol{R}_i$ is the residual (equilibrium gap) that it may effaced after j iterations:

$[\boldsymbol{K}ep].\delta\boldsymbol{u}=\boldsymbol{R}i\text{-}j$ (3)

When $\lim_{j\to\infty}\|\boldsymbol{R}_{i\text{-}j}\|=0$, then stability occurs. The physical meaning of this stability is that the system (body) is capable of carrying the loadings without failure. Failure modes are like rupture, sliding, slipping, overturning or large deformations. When eq. (3) fails to converge or it is diverging, the system becomes unstable. This instability means that a failure is overcoming. When [$\boldsymbol{K}ep$] is variable within the iterations a *Newton-Raphson* method is considered. When [$\boldsymbol{K}ep$] is constant within the iterations the method is called *modified Newton-Raphson*.

X-Fem solves such problems. The user of the software is monitoring the convergence process by the help of graphical representation of the $\|\boldsymbol{R}_{i\text{-}j}\|$. The criterion of convergence is $\|\boldsymbol{R}_{i\text{-}j}\|< R_{\min}$, $R_{\min}=a.\|\delta\boldsymbol{F}\|$. a is a value between 10^{-4} and 10^{-2} for the usual demands on accuracy/precision. When the convergence is asymptotic the process of iterations may stop if the deformations are small – negligible.

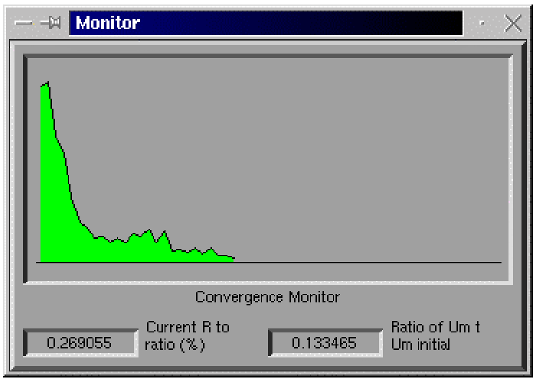

Fig.1 Monitoring of Convergence

2. SOME APPLICATIONS

Here is three application of X-Fem of nonlinear problems. The applications we consider are the elastic and inelastic buckling and the nonlinear analysis, the growth of the plastic boundary on a plate and the plastic boundary around a circular tunnel.

2.1 Buckling

Buckling is a phenomenon of instability developed on bars and thin plates. Compressed bars and thick plates are collapsing due to elastic instability without even considering inelastic behavior. The load that causes the collapse could be derived from the solution of an ordinary differential equation.

On the following figures there is represented the deformed state of two plates and the line-graphs of deformation-load for elastic and an elastoplastic buckling. These cases were calculated with X-Fem. In the elastoplastic analysis the elliptic – paraboloid criterion is considered.

To get solution for every deformed state, the simulation of the buckling is made with applied deformations (such a strain controlled experiment). A simulation with applied loads could not give solution after the critical load and the solution gets unstable.

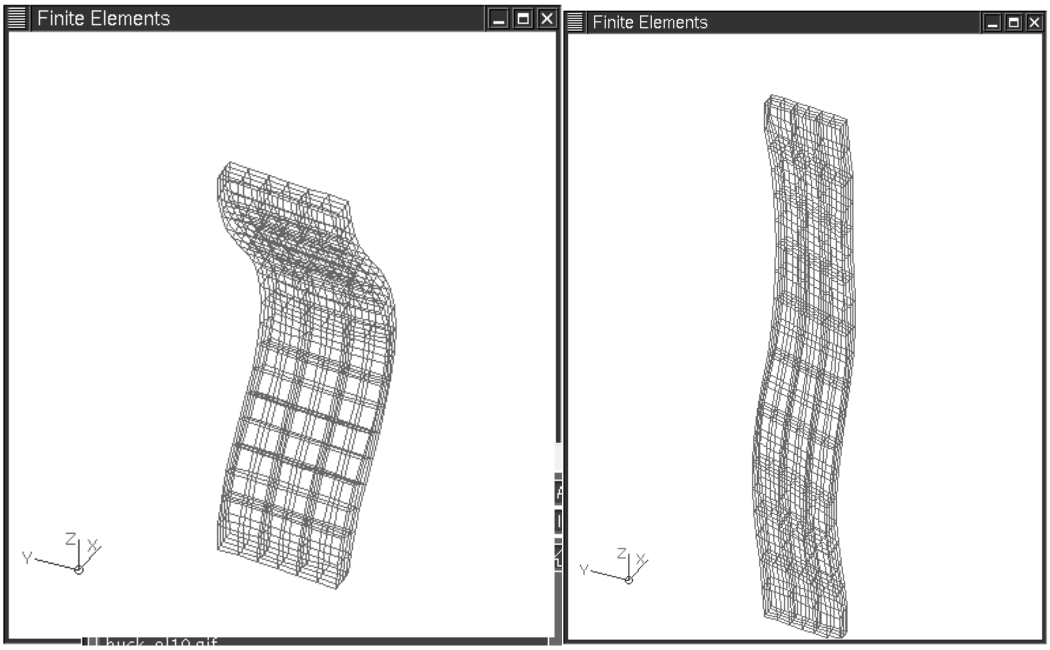

Fig. 2 Buckling of a thin plate (a) elastic, (b) elastoplastic

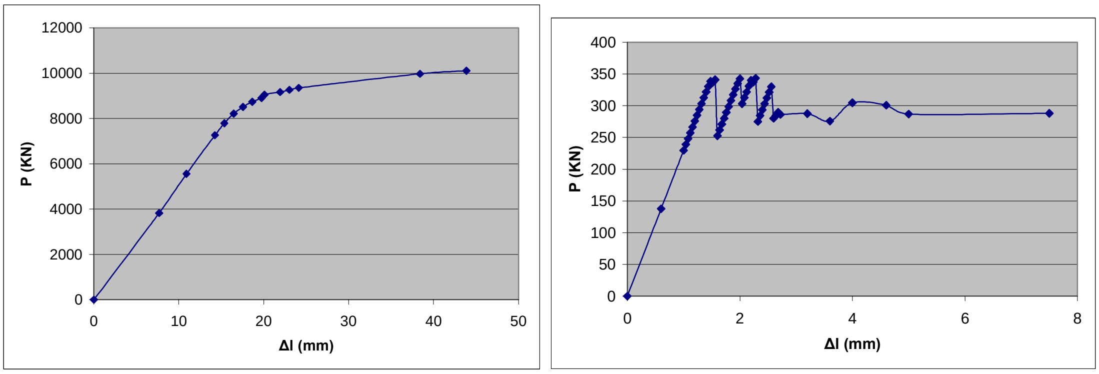

Fig.3 Instability caused by buckling (a) elastic, (b) elastoplastic

2.2 Growth of plastic boundary

Here are four stages of the growth of a plastic boundary on a plate with stress concentration caused by 45^o cuts. The diffusion of the plastic boundary has nonlinear nature. The problem is symmetric on both planes *xy* and *zx*. One quarter is showed on the following figures.

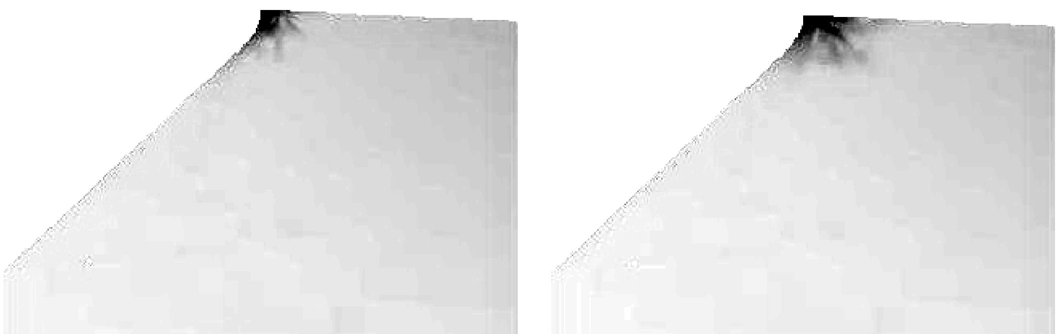

Fig. 4a Diffusion of the plastic boundary, stages 2 (a) and 4 (b)

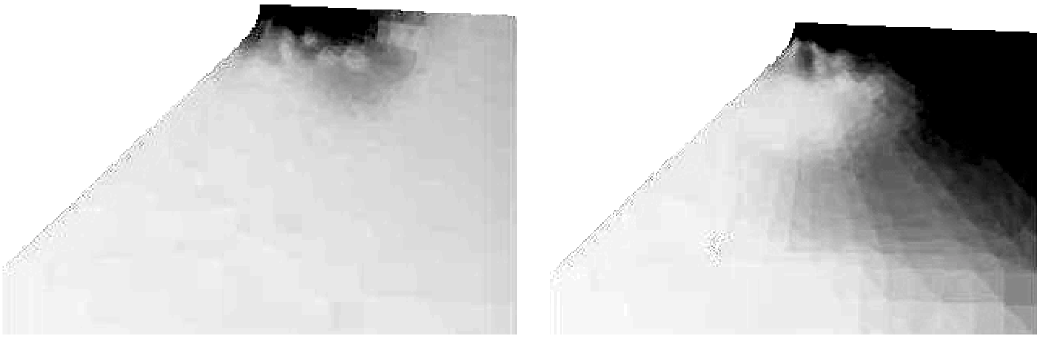

Fig. 4b Diffusion of the plastic boundary, stages 6 (a) and 8 (b)

2.3 Plastic Boundary around a tunnel opening

In the figure bellow, the plastic boundary around a tunnel is exposed. The plastic boundary diffusion around the hole has also nonlinear nature. The problem is symmetric on the plane *zx*. One half is showed.

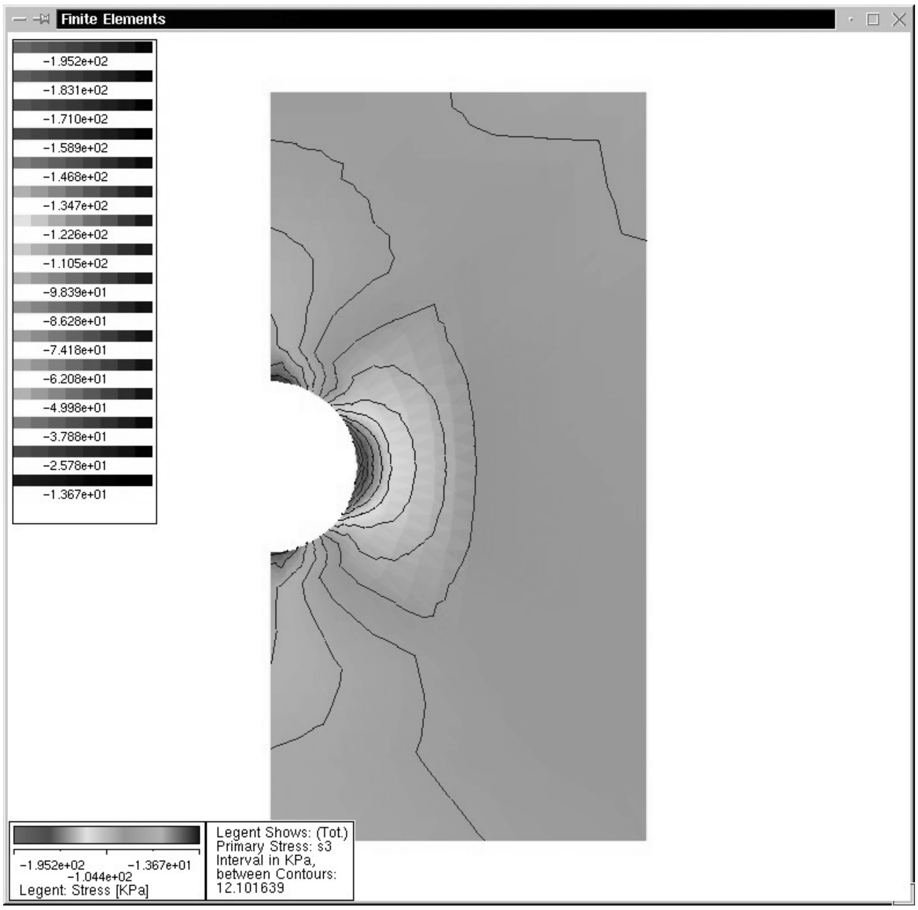

Fig. 5 Stress concentrations around a tunnel

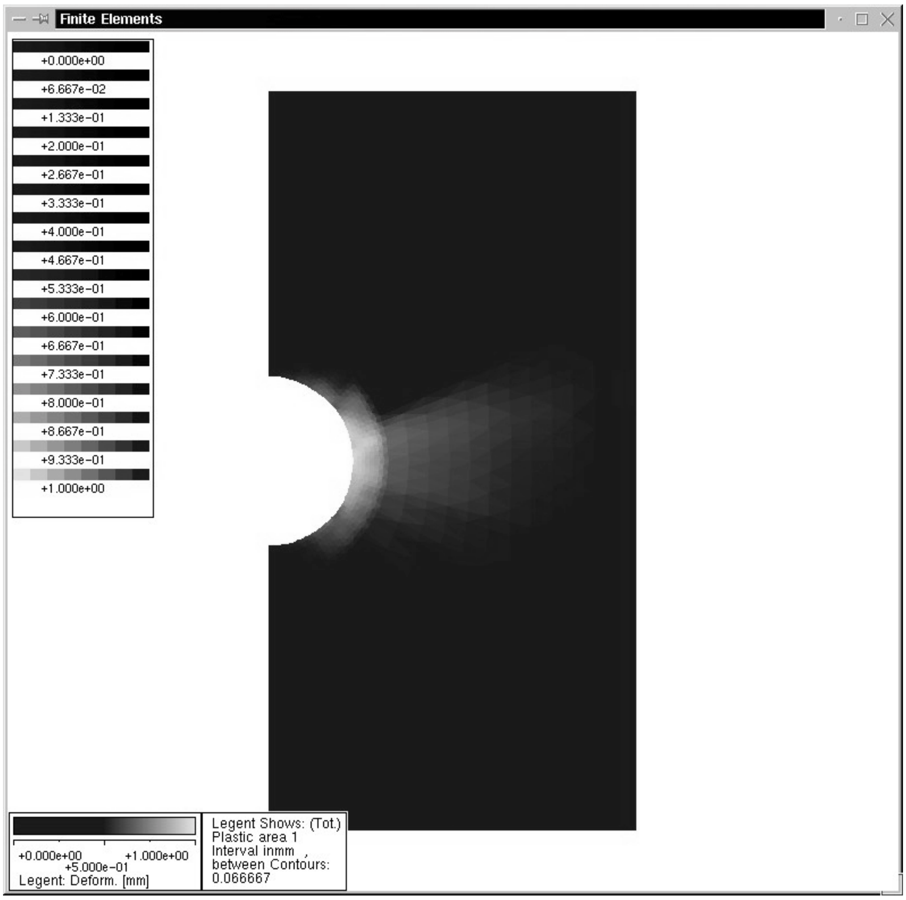

Fig. 6 Plastic boundary around a tunnel (white area)